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Deterministic identity testing paradigms for bounded top-fanin depth-4 circuits

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Abstract. Polynomial Identity Testing (PIT) is a fundamental computational problem, and the study of depth-4 circuits has been an enticing pursuit since the reduction result of Agrawal and Vinay (FOCS 2008). A restricted depth-4 circuit of the form $\sum_{i=1}^{k} \prod_j g_{ij}$, where $\deg g_{ij} \leq \delta$, is called a $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuit. By further restricting each g_{ij} to be a sum of univariates, we obtain $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits. These restricted models have seen significant attention over the last two decades. In this work, we solve two basic underlying open problems. We give the first polynomial-time PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits. We also give the first quasipolynomial-time *blackbox* PIT for both $\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits. A key technical ingredient in our algorithms is the use of the *logarithmic derivative* and its power series to modify the top Π-gate to a $\wedge$-gate.

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1 Introduction: PIT and beyond

Algebraic circuits are natural algebraic analogues of boolean circuits, with the logical operations being replaced by + and × operations over the underlying field. The study of algebraic circuits forms an integral part of the research carried out in algebraic complexity, mainly pioneered (and formalised) by Valiant [99]. A central problem in algebraic complexity is an algorithmic design problem, known as Polynomial Identity Testing (PIT): given an algebraic circuit C over a field $\mathbb{F}$ and input variables $x_1, \ldots, x_n$, determine whether C computes the identically zero polynomial. PIT has found numerous applications and connections to other algorithmic problems. Among the examples are algorithms for finding perfect matchings in graphs [67, 72, 28], primality testing [4], polynomial factoring [60, 24], polynomial equivalence [17], reconstruction algorithms [56, 94, 53] and the existence of algebraic natural proofs [16, 61]. Moreover, efficient design of PIT algorithms is intrinsically connected to proving strong lower bounds [48, 1, 50, 27, 33, 18, 25]. Interestingly, PIT also emerges in many fundamental results in complexity theory such as IP = PSPACE [93, 68], the PCP theorem [10, 11], and the overarching Geometric Complexity Theory (GCT) program towards P ≠ NP [73, 71, 39, 40].

There are broadly two settings in which the PIT question can be framed. In the *whitebox* setup, we are allowed to look inside the wirings of the circuit, while in the *blackbox* setting we can only evaluate the circuit at some points from the given domain. There is a very simple randomised algorithm for this problem — evaluate the polynomial at a random point from a large enough domain. With very high probability, a nonzero polynomial will have a nonzero evaluation; this is famously known as the Polynomial Identity Lemma [76, 19, 101, 92]. It has been a long standing open question to derandomise this algorithm.

For many years, blackbox identity tests were only known for depth-2 circuits which compute sparse polynomials [13, 57]. In a surprising result, Agrawal and Vinay [7] showed that a complete derandomisation of blackbox identity testing for just depth-4 algebraic circuits ($\Sigma\Pi\Sigma\Pi$) already implies a near complete derandomisation for the general PIT problem. More recent depth reduction results [58, 45], and the bootstrapping phenomenon [2, 62, 42, 9] show that even PIT for very restricted classes of depth-4 circuits (*even* depth-3) would have very interesting consequences for PIT of general circuits. These results make the identity testing of depth-4 circuits, a very meaningful pursuit.

Three PITs in one-shot. Following the same spirit, here we solve three important (and open) PIT questions. We give the first deterministic polynomial-time whitebox PIT algorithm for the bounded sum of product of sum of univariates circuits [81, Open Prob. 2]. Further, we give a quasipolynomial-time blackbox algorithm for the same class of circuits. These circuits are denoted by $\Sigma^{[k]}\Pi\Sigma\wedge$ and compute polynomials of the form $\Sigma_{i\in[k]}\Pi_j \left(g_{ij1}(x_1) + \cdots + g_{ijn}(x_n)\right)$.

> *Whitebox and Blackbox PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits is in polynomial and quasi-polynomial time respectively.*

A similar technique also gives a quasi-polynomial time blackbox PIT algorithm for circuits that have the form of a bounded sum of product of bounded degree sparse polynomials. They are denoted by $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ (where k and δ can be up to $\text{poly}(\log(s))$, where s is the circuit size).

Blackbox PIT for the $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits is in quasi-polynomial time.

$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits compute polynomials which are of the form $\Sigma_{i\in[k]}\Pi_j g_{ij}(\boldsymbol{x})$, where $\deg(g_{ij}) \leq \delta$. Even $\delta = 2$ was a challenging open problem [63, Open Problem 2]. The model has gained a lot of interest in the past few years and has generated many important results [78, 79, 35, 75, 43, 36].

1.1 Main results: An analytic detour to three PITs

Though some attempts have been made to solve PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$, even in the whitebox settings, an efficient PIT for $k \geq 3$ *even* remains open, see [81, Open Prob. 2]. Our first result addresses this problem and designs a polynomial time algorithm (Refer Algorithm 1). In our pursuit we discover an analytic and non-ideal based new technique which we refer to as DiDI. Throughout the paper, we will work with $\mathbb{F} = \mathbb{Q}$, though all the results hold for fields of large characteristic.

Theorem 1.1 (Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$ PIT)**.** *There is a deterministic, whitebox $s^{O(k\,7^k)}$-time PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits of size s, over $\mathbb{F}[\boldsymbol{x}]$.*

Remark 1.2.

1. Case $k \leq 2$ can be solved by invoking [81, Theorem 5.2]; but $k \geq 3$ was open.
2. Our technique *necessarily* blows up the exponent exponentially in k. In particular, it would be interesting to design an efficient time algorithm when $k = \Theta(\log s)$.
3. It is not clear if the current technique gives PIT for $\Sigma^{[k]}\Pi\Sigma\mathsf{M}_2$ circuits, where $\Sigma\mathsf{M}_2$ denotes sum of *bi*variate monomials computed and fed into the top product gate.

Next, we go to the blackbox setting and address two models of interest, namely— $\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$, where k, δ are constants. Our work builds on previous ideas for unbounded top fanin (1) Jacobian [5], (2) the known blackbox PIT for $\Sigma\wedge\Sigma\wedge$ and $\Sigma\wedge\Sigma\Pi^{[\delta]}$ [46, 29] while maneuvering with an analytic approach *via* power-series, which unexpectedly *reduces* the top Π-gate to a $\wedge$-gate.

Theorem 1.3 (Blackbox depth-4 PIT)**.**

(a) There is a $s^{O(k\log\log s)}$ time blackbox PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits of size s, over $\mathbb{F}[\boldsymbol{x}]$.

(b) There is a $s^{O(\delta^2 k\,\log s)}$ time blackbox PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits of size s, over $\mathbb{F}[\boldsymbol{x}]$.

Remark 1.4.

1. Theorem 1.3 (b) has a *better* dependence on k, but *worse* on s, than Theorem 1.1. Our results are quasipoly-time even up to $k, \delta = \text{poly}(\log s)$.
2. Theorem 1.3 (a) is better than Theorem 1.3 (b), because $\Sigma\wedge\Sigma\wedge$ has a faster algorithm than $\Sigma\wedge\Sigma\Pi^{[\delta]}$.
3. Even for $\Sigma^{[3]}\Pi\Sigma\wedge$ and $\Sigma^{[3]}\Pi\Sigma\Pi^{[3]}$ models, we leave the *poly*-time blackbox question open.

1.2 Prior works on related models

In the last two decades, there has been a surge of results on identity testing for restricted classes of bounded depth algebraic circuits (for example, 'locally' bounded independence, bounded read/occur, bounded variables). For comprehensive surveys of these developments, we refer the reader to [87, 88, 22]. Here, we briefly highlight the results most relevant to our models.

There have been numerous results on PIT for depth-3 circuits with bounded top fanin (known as $\Sigma^{[k]}\Pi\Sigma$-circuits). Dvir and Shpilka [26] gave the first quasipolynomial-time deterministic whitebox algorithm for $k = O(1)$, using rank based methods, which finally led Karnin and Shpilka [52] to design an algorithm of same complexity in the blackbox setting. Kayal and Saxena [55] gave the first polynomial-time algorithm for this model. Later, a series of works in [89, 90, 91, 5] generalised the model and gave $n^{O(k)}$-time algorithm when the algebraic rank of the product polynomials are bounded. Note that in the white-box setting, our algorithm gives a poly(s)-time PIT algorithm for bounded top-fanin depth-3 circuits. Moreover, the dependence on the top-fanin is exponential. In the blackbox setting, our algorithm solves PIT for bounded top-fanin depth-3 circuits in quasi-poly(s) time, hence it does not offer any speedup compared to known polynomial time algorithms. However, our algorithm does give a PIT idea that is different from the known ones.

There has also been some progress on PIT for restricted classes of depth-4 circuits. A quasipolynomial-time blackbox PIT algorithm for *multilinear* $\Sigma^{[k]}\Pi\Sigma\Pi$-circuits was designed in [51], which was further improved to a $n^{O(k^2)}$-time deterministic algorithm in [85]. A quasipolynomial blackbox PIT was given in [12, 63] when algebraic rank of the irreducible factors in each multiplication gate as well as the bottom fanin are bounded. Further interesting restrictions like sum of product of fewer variables, and more structural restrictions have been exploited, see [32, 6, 29, 70, 64]. Some progress has also been made for bounded top-fanin and bottom-fanin depth-4 circuits via incidence geometry [44, 95, 78].

Garg, Oliveira, and Sengupta [36] gave the first polynomial-time PIT algorithm for $\Sigma^{[3]}\Pi\Sigma\Pi^{[d]}$-circuits with constant d, using non-linear Edelstein-Kelly theorem (see also [78]). In an independent and parallel work, Guo and Wang [43] presented a polynomial-time PIT algorithm for the same model under the additional restriction that one of the top summands is square-free, notably avoiding the use of Sylvester-Gallai-type theorems.

The authors recently generalised their novel DiDI-technique to solve 'border PIT' of depth-4 circuits [20]. Specifically, they give a $s^{O(k\cdot 7^k\cdot \log log s)}$ time and $s^{O(\delta^2\cdot k\cdot 7^k\cdot \log s)}$ time blackbox PIT algorithm for $\overline{\Sigma^{[k]}\Pi\Sigma\wedge}$ and $\overline{\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}}$ respectively. By definition, border classes capture exact complexity classes, hence border PIT results seemingly subsume the results we present in this paper. However, the whitebox PIT algorithm here is much more efficient than their quasi-poly time blackbox algorithm. Further, the time complexity of blackbox PIT algorithms has a better dependence on k and δ compared to their exponential dependence. Lastly, the proofs in this paper are simpler as we don't have to deal with an infinitesimally close approximation of polynomials in border complexity classes. Very recently, Dutta and Saxena [23] showed an exponential-gap fanin-hierarchy theorem for bounded depth-3 circuits which is also based on a *finer* generalisation of the DiDI-technique.

Model	Time	Ref.
$\Sigma^{[k]}\Pi^{[d]}\Sigma$	poly(n, d^k)	[90]
Multilinear $\Sigma^{[k]}\Pi\Sigma\Pi$	poly$(n^{O(k^2)})$	[85, 5]
$\Sigma\Pi\Sigma\Pi$ of bounded trdeg	poly(s^{trdeg})	[12]
$\Sigma^{(k)}\Pi\Sigma\Pi^{[d]}$ of bounded *local* trdeg	quasi-poly(n)	[64]
$\Sigma^{[3]}\Pi\Sigma\Pi^{[O(1)]}$	poly(n, d)	[79, 36]
$\overline{\Sigma^{[k]}\Pi\Sigma\wedge}$	$s^{O(k\cdot 7^k\cdot \log log s)}$	[20]
$\overline{\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}}$	$s^{O(\delta^2\cdot k\cdot 7^k\cdot \log s)}$	[20]
$\Sigma\Pi\Sigma\Pi$	SUBEXP(n)	[66]
Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$	$s^{O(k\,7^k)}$	This work.
$\Sigma^{[k]}\Pi\Sigma\wedge$	$s^{O(k \log\log s)}$	This work.
$\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$	$s^{O(\delta^2\, k\, \log s)}$	This work.

Table 1: Time complexity comparison of PIT algorithms related to $\Sigma\Pi\Sigma\Pi$ circuits

In a breakthrough result by Limaye, Srinivasan and Tavenas [66] the *first* superpolynomial lower bound for constant depth circuits was obtained. Their lower bound result, together with the 'hardness vs randomness' tradeoff result of [18] gives the *first* deterministic blackbox PIT algorithm for general depth-4 circuits which runs in $s^{O(n^\epsilon)}$ for all real $\epsilon > 0$. Their result is the first *sub*exponential time PIT algorithm for depth-4 circuits. Moreover, compared to their algorithm, our quasipoly time blackbox and polynomial time whitebox algorithms are significantly faster.

Limitations of known techniques. People have studied depth-4 PIT only with extra restrictions, mostly due to the limited applicability of the existing techniques as they were tailor-made for the specific models and do not generalise. For example, the previous methods handle $\delta = 1$ (that is, linear polynomials at the bottom) or $k = 2$ (via *factoring*, [81]). While $k = 2$ to 3, or $\delta = 1$ to 2 (that is, 'linear' to 'quadratic') already demands a qualitatively different approach.

Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$ model generalises the famous bounded top fanin depth-3 circuits $\Sigma^{[k]}\Pi\Sigma$ of [55]; but their Chinese Remaindering (CR) method, loses applicability and thus fails to solve even a slightly more general model. The blackbox setting involved similar 'certifying path' ideas in [90] which could be thought of as general CR. It comes up with an ideal I such that $f \neq 0 \bmod I$ and finally preserves it under a constant-variate linear map. The preservation gets harder (for both $\Sigma^{[k]}\Pi\Sigma\wedge$ and $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$) due to the increased non-linearity of the generators of ideal I. Intuitively, larger δ via ideal-based routes, brings us to the Gröbner basis method

(which is doubly-exponential-time in n) [100]. We know that ideals even with 3-generators (analogously $k = 4$) already capture the whole ideal-membership problem [83].

The algebraic-geometric approach to tackle $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ has been explored in [12, 44, 70, 41]. The families which satisfy a certain Sylvester–Gallai configuration (called SG-circuits) is the harder case which is conjectured to have constant transcendence degree [44, Conj. 1]. Non-SG circuits is the case where the nonzeroness-certifying-path question reduces to radical-ideal non-membership questions [37]. This is really a variety question where one could use algebraic-geometry tools to design a poly-time blackbox PIT. In fact, very recently, Guo [41] gave a s^{δ^k}-time PIT by constructing explicit variety evasive subspace families. Unfortunately, this is not the case in the ideal non-membership; this scenario makes it much harder to solve $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$. From this viewpoint, radical-ideal-membership explains well why the intuitive $\Sigma^{[k]}\Pi\Sigma$ methods do not extend to $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$.

Interestingly, Forbes [29] gave a quasipolynomial-time PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$ using shifted-partial derivative techniques; but it naively fails when one replaces the $\wedge$-gate by Π (because the 'measure' becomes too large). Our blackbox PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ crucially relies on the hitting set for $\Sigma\wedge\Sigma\Pi^{[\delta]}$. While Shpilka and Tal [96] recently achieved a polynomial-time PIT for this depth-4 powering model, their requirement that the powering degree be at least the square of the top fanin precludes any improvement to our $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ result. The duality trick of [86] completely solves whitebox PIT for $\Sigma\wedge\Sigma\wedge$, by transforming it to a read-once oblivious ABP (ROABP); but it is inapplicable to our models which have a product gate (Π) at the top (due to large waring rank and ROABP-width). A priori, our models are incomparable to ROABP, and thus the well-known PIT algorithms for ROABP [32, 31, 46] are not expected to help either.

Similarly, a naive application of the *Jacobian* and *certifying path* technique from [5] fails for our models because it is difficult to come up with a *faithful* map for constant-variate reduction. Kumar and Saraf [63] crucially used that the computed polynomial has low individual degree (such that [27] can be invoked), while in [64] they exploited the low algebraic rank of the polynomials computed below the top Π-gate. Neither of them holds in general for our models. Very recently, Peleg and Shpilka [79] gave a poly-time blackbox PIT for $\Sigma^{[3]}\Pi\Sigma\Pi^{[2]}$, via incidence geometry (for example, Edelstein-Kelly theorem involving 'quadratic' polynomials), by solving [44, Conj. 1] for $k = 3, \delta = 2$. The method seems very strenuous to generalise even to 'cubic' polynomials ($\delta = 3 = k$).

PIT for other models. Blackbox PIT algorithms for many restricted models are known. for example, ROABP related models [80, 49, 3, 46, 47, 31, 8], log-variate circuits [30, 14], and non-commutative models [38, 65].

1.3 Techniques and motivation

Both the proofs are analytic as they use *logarithmic derivative*, and its power-series expansion which greatly transform the respective models. Where the nature of the first proof is inductive, the second is a more direct *one-shot* proof. In both the cases, we essentially reduce to the better-understood *wedge* models, that have unbounded top fanin, yet for which PITs are known. As we will describe it further, this reduction was unforeseeable and quite powerful.

The analytic tool that we use, appears in algebra and complexity theory through the *formal power series* ring $\mathsf{R}[\![x_1, \ldots, x_n]\!]$ (in short $\mathsf{R}[\![\boldsymbol{x}]\!]$), see [74, 98, 24]. The advantages of the ring $\mathsf{R}[\![\boldsymbol{x}]\!]$ are many and they usually emerge because of the inverse identity: $(1 - x_1)^{-1} = \sum_{i \geq 0} x_1^i$, which does not make sense in $\mathsf{R}[x]$, but is valid in $\mathsf{R}[\![\boldsymbol{x}]\!]$. Other analytic tools used are inspired from Wronskian (linear dependence) [59, Theorem 7] [54], Jacobian (algebraic dependence) [12, 5, 77], and logarithmic derivative operator $\mathrm{dlog}_{z_1}(f) = (\partial_{z_1} f)/f$.

We will work with the division operator (for example, $\mathsf{R}(z_1)$, over a certain ring R). However, the divisions do not come for free as they require invertibility with respect to z_1 throughout (again landing us in $\mathsf{R}[\![z_1]\!]$). For circuit classes $\mathcal{C}, \mathcal{D}$ we define class

$$\mathcal{C}/\mathcal{D} := \{f/g \mid f \in \mathcal{C}, \mathcal{D} \ni g \neq 0\}.$$

Similarly $\mathcal{C} \cdot \mathcal{D}$ denotes the class taking respective products.

1.3.1 The DiDI-technique

In Theorem 1.1 we introduce a novel technique for designing PIT algorithms which consists of inductively applying two fundamental operations on the input circuits to reduce them to a more tractable model. Suppose we want to test $\sum_{i \in [k]} T_i \stackrel{?}{=} 0$ where each T_i is computable by $\Pi\Sigma\wedge$. The idea is to *DI*vide it by T_k to obtain $1 + \sum_{i \in [k-1]} T_i/T_k$ and then apply the *D*erivative to reduce the fanin to $k - 1$ and obtain $\sum_{i \in [k-1]} \mathcal{T}_i$. Naturally, these operations push us to work with the fractional ring (for example, $\mathsf{R}(z_1)$, over a certain ring R), further it also distorts the model as $\mathcal{T}_i$'s are no longer computable by simple $\Pi\Sigma\wedge$ circuits. However, with careful analytical analysis we establish that the non-zeroness is preserved in the reduced model. The process is then repeated until we reach $k = 1$, while maintaining the invariants which help us in preserving the non-zeroness till the end. We finish the proof by showing that the identity testing of the reduced model can be done using known PIT algorithms.

1.3.2 Jacobian hits again

In Theorem 1.3 we exploit the prowess of the Jacobian polynomial first introduced in [12] and later explored in [5] to unify known PIT algorithms and design new ones. Suppose we want to test $\sum_{i \in [k]} T_i \stackrel{?}{=} 0$, where $T_i \in \Pi\Sigma\Pi^{[\delta]}$ (respec. $\Pi\Sigma\wedge$). We associate the Jacobian $J(T_1, \ldots, T_r)$ to capture the algebraic independence of $T_1, \ldots, T_r$ assuming these to be a transcendence basis of the T_i's. We design a variable-reducing linear map Φ which preserves the algebraic independence of $T_1, \ldots, T_r$ and show that for any C: $C(T_1, \ldots, T_k) = 0 \iff C(\Phi(T_1), \ldots, \Phi(T_k)) = 0$. Such a map is called 'faithful' [5]. The map Φ ultimately provides a hitting set for $T_1 + \ldots + T_k$, as we reduce to a PIT of a polynomial over 'few' (roughly equal to k) variables, yielding a quasi-polynomial time algorithm.

2 Preliminaries

Before proving the results, we describe some of the assumptions and notations used throughout the paper. $\boldsymbol{x}$ denotes $(x_1, \ldots, x_n)$. $[n]$ denotes $\{1, \ldots, n\}$.

2.1 Notations and Definitions

- **Logarithmic derivative.** Over a ring R and a variable y, the logarithmic derivative $\mathrm{dlog}_y : \mathsf{R}(y) \to \mathsf{R}(y)$ is defined as $\mathrm{dlog}_y(f) := \partial_y f/f$; here ∂_y denotes the partial derivative with respect to variable y. One important property of dlog is that it is additive over a product as

$$\mathrm{dlog}_y(f \cdot g) = \frac{\partial_y(f \cdot g)}{f \cdot g} = \frac{(f \cdot \partial_y g + g \cdot \partial_y f)}{f \cdot g} = \mathrm{dlog}_y(f) + \mathrm{dlog}_y(g).$$

  We refer to this effect as the *linearization* of products.

- **Circuit size.** Sparsity $\mathrm{sp}(\cdot)$ refers to the number of nonzero monomials. In this paper, it is a parameter of the circuit size. In particular, $\mathrm{size}(g_1 \cdots g_s) = \sum_{i\in[s]} (\mathrm{sp}(g_i) + \deg(g_i))$, for $g_i \in \Sigma\wedge$ (respectively $\Sigma\Pi^{[\delta]}$). In the whitebox setting, we also include the *bit-complexity* of the circuit (that is, the bit complexity of the constants used in the wires) in the size parameter. Some of the complexity parameters of a circuit are *depth* (number of layers), *syntactic degree* (the maximum degree polynomial computed by any node), *fanin* (maximum number of inputs to a node).

- **Hitting set.** A set of points $\mathcal{H} \subseteq \mathbb{F}^n$ is called a *hitting-set* for a class C of n-variate polynomials if for any nonzero polynomial $f \in C$, there exists a point in $\mathcal{H}$ where f evaluates to a nonzero value. A $T(n)$-time hitting-set would mean that the hitting-set can be generated in time $T(n)$, for input size n.

- **Valuation.** Valuation is a map $\mathrm{val}_y : \mathsf{R}[y] \to \mathbb{Z}_{\geq 0}$, over a ring R, such that $\mathrm{val}_y(\cdot)$ is defined to be the maximum power of y dividing the element. It can be easily extended to fraction field $\mathsf{R}(y)$, by defining $\mathrm{val}_y(p/q) := \mathrm{val}_y(p) - \mathrm{val}_y(q)$; where it can be negative.

- **Field.** We denote the underlying field as $\mathbb{F}$ and assume that it is of characteristic 0. All our results hold for other fields (for example, $\mathbb{Q}_p, \mathbb{F}_p$) of *large* characteristic (see Remarks in Section 3–4).

- **Jacobian.** The Jacobian of a set of polynomials $\boldsymbol{f} = \{f_1, \ldots, f_m\}$ in $\mathbb{F}[\boldsymbol{x}]$ is defined to be the matrix $\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{f}) := \left(\partial_{x_j}(f_i)\right)_{m\times n}$. Let $S \subseteq \boldsymbol{x} = \{x_1, \ldots, x_n\}$ and $|S| = m$. Then, polynomial $J_S(\boldsymbol{f})$ denotes the minor (that is, determinant of the submatrix) of $\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{f})$, formed by the columns corresponding to the variables in S.

2.2 Basics of Algebraic Complexity Theory

For a detailed discussion on the basics of Algebraic Complexity Theory, we encourage readers to refer to [87, 97, 69, 88, 84, 22]. Here we will formally state a few of the PIT results and properties of circuits for the later reference.

Trivial PIT Algorithm

The simplest PIT algorithm for any circuit in general is due to the Polynomial Identity Lemma [76, 19, 101, 92]. When the number of variables is small, say $O(1)$, then this algorithm is very efficient.

Lemma 2.1 (Trivial PIT)**.** *For any n-variate, individual degree $< d$ polynomial $f \in \mathbb{F}[\boldsymbol{x}]$, there exists a deterministic PIT algorithm which runs in time $O(d^n)$.*

Sparse Polynomial

Sparse PIT is testing the identity of polynomials with a bounded number of monomials. There has been a lot of work on sparse-PIT; interested readers can refer to [13, 57] and the references therein. For the proof of poly-time hitting set of Sparse PIT see [87, Thm. 2.1].

Theorem 2.2 (Sparse-PIT map [57])**.** *Let $p(\boldsymbol{x}) \in \mathbb{F}[\boldsymbol{x}]$ with individual degree at most d and sparsity at most m. Then, there exists $1 \leq r \leq (mn \log d)^2$, such that*

$$p(y, y^d, \ldots, y^{d^{n-1}}) \neq 0, \bmod y^r - 1.$$

If p is computable by a size-s $\Sigma\Pi$ circuit, then there is a deterministic algorithm to test its identity which runs in time $\mathrm{poly}(s, m)$.

Indeed, if the identity of sparse polynomials can be tested efficiently, products of sparse polynomials can also be tested efficiently. We formalise this in the following:

Lemma 2.3 ([82] Lemma 2.3)**.** *For any n-variate, degree d polynomial $f \in \mathbb{F}[\boldsymbol{x}]$ computable by a $\Pi\Sigma\Pi$ of size s, there is a deterministic PIT algorithm which runs in time* $\mathrm{poly}(s, d)$.

A set $\mathcal{H} \subseteq \mathbb{F}^n$ is called a Hitting Set for a class of polynomials $\mathcal{C} \subseteq \mathbb{F}[\boldsymbol{x}]$, if for all $g \in \mathcal{C}$

$$g \neq 0 \iff \exists \boldsymbol{\alpha} \in \mathcal{H} : g(\boldsymbol{\alpha}) \neq 0.$$

In the literature, PIT has a close association with Hitting Sets as the two notions are provably equivalent (refer Lemma 3.2.9 and 3.2.10 [34]). Note that the set $\mathcal{H}$ works for every polynomial of the class. Instead of a PIT algorithm, we will occasionally use such a set.

Lemma 2.4 (Hitting Set of $\Pi\Sigma\wedge$)**.** *For any n-variate, degree d polynomial $f \in \mathbb{F}[\boldsymbol{x}]$ computable by a $\Pi\Sigma\Pi$ of size s, there is an explicit Hitting Set of size* $\mathrm{poly}(s, d)$.

Algebraic Branching Program (ABP)

An ABP is a layered directed acyclic graph with $q+1$ many layers of vertices $V_0, \ldots, V_q$ with a source a and a sink b such that all the edges in the graph only go from a to V_0, V_{i-1} to V_i for any $i \in [q]$, and V_q to b. The edges have *uni*variate polynomials as their weights. The ABP is said to compute the polynomial

$$f(\boldsymbol{x}) \;=\; \sum_{p \in \mathsf{paths}(a,b)} \prod_{e \in p} W(e)\,,$$

where $W(e)$ is the weight of the edge e. The ABP has width-w if $|V_i| \leq w$, $\forall i \in \{0, \ldots, q\}$. In an equivalent definition, polynomials computed by ABP are of the form $A^T(\prod_{i\in[q]} D_i)B$, where $A, B \in \mathbb{F}^{w\times 1}[\boldsymbol{x}]$, and $D_i \in \mathbb{F}^{w\times w}[\boldsymbol{x}]$, where entries are univariate polynomials. We encourage interested readers to refer [97, 69] for more detailed discussion.

Definition 2.5 (Read-once oblivious ABP (ROABP))**.** An ABP is called a *read-once oblivious ABP (ROABP)* if the edge weights are univariate polynomials in *distinct* variables across layers. Formally, there is a permutation π on the set $[q]$ such that the entries in the i-th matrix D_i are univariate polynomials over the variable $x_{\pi(i)}$, that is, they come from the polynomial ring $\mathbb{F}[x_{\pi(i)}]$.

A polynomial $f(x)$ is said to be computed by width-w ROABPs in *any order*, if for every permutation σ of the variables, there exists a width-w ROABP in the variable order σ that computes the polynomial $f(\boldsymbol{x})$. In the whitebox setting, the identity testing of any-order ROABPs is completely solved.

Theorem 2.6 (Theorem 2.4 [80])**.** *For n-variate polynomials computed by size-s ROABP, a hitting set of size $O(s^5 + s \cdot n^4)$ can be constructed.*

There have been quite a few results on blackbox PIT for ROABPs as well [32, 31, 46]. The current best known algorithm works in quasipolynomial time.

Theorem 2.7 (Theorem 4.9 [46])**.** *For n-variate, individual-degree-d polynomials computed by width-w ROABPs in any order, a hitting set of size $(ndw)^{O(\log\log w)}$ can be constructed.*

Depth-4 Circuits

A polynomial $f(\boldsymbol{x}) \in \mathbb{F}[\boldsymbol{x}]$ is computable by $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits if $f(\boldsymbol{x}) = \sum_{i\in[s]} f_i(\boldsymbol{x})^{e_i}$ where $\deg f_i \leq \delta$. The first nontrivial PIT algorithm for this model was designed in [29].

Theorem 2.8 (Proposition 4.18 [29])**.** *There is a $\mathrm{poly}(n, d, \delta\log s)$-explicit hitting set of size $(nd)^{O(\delta\log s)}$ for the class of n-variate, degree-$(\leq d)$ polynomials $f(\boldsymbol{x})$, computed by $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size s.*

Similarly, $\Sigma\wedge\Sigma\wedge$ circuits compute polynomials of the form $f(\boldsymbol{x}) = \sum_{i\in[s]} f_i^{e_i}$ where f_i is a sum of univariate polynomials. Using the duality trick [86] and PIT results from [80, 46], one can design an efficient PIT algorithm for $\Sigma\wedge\Sigma\wedge$ circuits.

Lemma 2.9 (PIT for $\Sigma\wedge\Sigma\wedge$-circuits)**.** *Let $P \in \mathbb{F}[\boldsymbol{x}]$ be computable by a $\Sigma\wedge\Sigma\wedge$ circuit of size s. Then, there exists a* poly(s) *(respectively $s^{O(\log\log s)}$) time whitebox (respectively blackbox) PIT for the same.*

Proof sketch. We show that any $g(\boldsymbol{x})^e = (g_1(x_1) + \ldots + g_n(x_n))^e$, where $\deg(g_i) \leq s$ can be written as $\sum_j h_{j1}(x_1)\cdots h_{jn}(x_n)$, for some $h_{j\ell} \in \mathbb{F}[x_\ell]$ of degree at most es. Define, $G := (y+g_1)\cdots(y+g_n) - y^n$. In its e-th power, notice that the leading-coefficient is $\text{coef}_{y^{e(n-1)}}(G^e) = g^e$. So, interpolate on $e(n-1)+1$ many points ($y = \beta_i \in \mathbb{F}$) to get

$$\text{coef}_{y^{e(n-1)}}(G^e) \;=\; \sum_{i=1}^{e(n-1)+1} \alpha_i\, G^e(\beta_i)\;.$$

Now, expand $G^e(\beta_i) = ((\beta_i+g_1)\cdots(\beta_i+g_n) - \beta_i^n)^e$, by binomial expansion (without expanding the inner n-fold product). The top-fanin can be at most $s\cdot(e+1)\cdot(e(n-1)+1) = O(se^2n)$. The individual degrees of the intermediate univariates can be at most es. Thus, it can be computed by an ROABP (of *any order*) of size at most $O(s^2e^3n)$.

Now, if $f = \sum_{j\in[s]} f_j^{e_j}$ is computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s, then clearly, f can also be computed by an ROABP (of any order) of size at most $O(s^6)$. So, the whitebox PIT follows from Theorem 2.6, while the blackbox PIT follows from Theorem Theorem 2.7. □

Further, $\Sigma\wedge\Sigma\wedge$ can be shown to be closed under multiplication. That is, the product of two polynomials, each computable by a $\Sigma\wedge\Sigma\wedge$ circuit, is computable by a single $\Sigma\wedge\Sigma\wedge$ circuit. To prove this, we use interpolation to write a product of a few powers as a sum of powers. For an algebraic proof, see [15, Proposition 4.3].

Lemma 2.10 (Waring Identity for a monomial)**.** *Let $M = x_1^{b_1}\cdots x_k^{b_k}$, where $1 \leq b_1 \leq \ldots \leq b_k$, and* roots of unity $\mathcal{Z}(i) := \{z \in \mathbb{C} : z^{b_i+1} = 1\}$. *Then,*

$$M = \sum_{\varepsilon(i)\in\mathcal{Z}(i):i=2,\cdots,k} \gamma_{\varepsilon(2),\ldots,\varepsilon(k)}\cdot(x_1 + \varepsilon(2)x_2 + \ldots + \varepsilon(k)x_k)^d\;,$$

where $d := \deg(M) = b_1 + \ldots + b_k$, and $\gamma_{\varepsilon(2),\ldots,\varepsilon(k)}$ are scalars ($\text{rk}(M) := \prod_{i=2}^k (b_i+1)$ *many).*

Remark. We actually need not work with $\mathbb{F} = \mathbb{C}$. We can go to a small extension (at most d^k), for a monomial of degree d, to make sure that $\varepsilon(i)$ exists.

Using the above lemma we prove the closure result.

Lemma 2.11. *Let $f_i(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$, of syntactic degree $\leq d_i$, be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s_i, for $i \in [k]$ (wrt $\boldsymbol{x}$). Then, $f_1\cdots f_k$ has $\Sigma\wedge\Sigma\wedge$ circuit of size $O((d_2+1)\cdots(d_k+1)\cdot\, s_1\cdots s_k)$.*

Proof. Let $f_i = \sum_j f_{ij}^{e_{ij}}$; by assumption $e_{ij} \leq d_i$ (by assumption). Then using Lemma 2.10, $f_{1j_1}^{e_{1j_1}}\cdots f_{kj_k}^{e_{kj_k}}$ has size at most $(d_2+1)\cdots(d_k+1)\cdot\left(\sum_{i\in[k]} \text{size}(f_{ij_i})\right)$, for indices $j_1,\ldots,j_k$. Summing up for all $s_1\cdots s_k$ many products (at most) gives the upper bound. □

3 Whitebox PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$

We consider a bloated model of computation which naturally generalises $\Sigma\Pi\Sigma\wedge$ circuits and works ideally under the DiDI-techniques.

Definition 3.1. We call a circuit $C \in \mathsf{Gen}(k, s)$, over $\mathsf{R}(\boldsymbol{x})$, for any ring R, with parameter k and size-s, if $C \in \Sigma^{[k]}(\Pi\Sigma \wedge /\Pi\Sigma\wedge) \cdot (\Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge)$. It computes $f \in \mathsf{R}(\boldsymbol{x})$, if $f = \sum_{i=1}^{k} T_i$, where

- $T_i =: (U_i/V_i) \cdot (P_i/Q_i)$, for $U_i, V_i \in \Pi\Sigma\wedge$, and $P_i, Q_i \in \Sigma\wedge\Sigma\wedge$,
- $\text{size}(T_i) = \text{size}(U_i) + \text{size}(V_i) + \text{size}(P_i) + \text{size}(Q_i)$, and $\text{size}(f) = \sum_{i\in[k]} \text{size}(T_i)$.

It is easy to see that all size-s $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits are in $\mathsf{Gen}(k, s)$. We will design the *recursive* algorithm on $\mathsf{Gen}(k, s)$.

Theorem 1.1 (Whitebox $\Sigma^{[k]}\Pi\Sigma\wedge$ PIT)**.** *There is a deterministic, whitebox $s^{O(k\,7^k)}$-time PIT algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$ circuits of size s, over $\mathbb{F}[\boldsymbol{x}]$.*

Proof. Begin with defining $T_{i,0} := T_i$ and $f_0 := f$ where $T_{i,0} \in \Pi\Sigma\wedge$; $\sum_i T_{i,0} = f_0$, and f_0 has size $\leq s$. Assume $\deg(f) < d \leq s$; we keep the parameter d separately, to help optimize the complexity later. In every recursive call we work with $\mathsf{Gen}(\cdot,\cdot)$ circuits.

As the input case, define $U_{i,0} := T_{i,0}$ and $V_{i,0} := P_{i,0} := Q_{i,0} := 1$. We will use the hitting set of product of sparse polynomials (refer Section 2.2) to obtain a point $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{F}^n$ such that $U_{i,0}|_{\boldsymbol{x}=\boldsymbol{\alpha}} \neq 0$, for all $i \in [k]$. Eventually this evaluation point will help in maintaining the invertibility of $\Pi\Sigma\wedge$. Consider

$$g := \prod_{i\in[k]} T_{i,0} = \prod_{i\in[k]} U_{i,0} = \prod_{i\in[\ell]} \sum_{j\in[n]} f_{ij}(x_j)\,,$$

where $f_{ij}(x_j)$ are univariate polynomials of degree at most d and $\ell \leq k\cdot s$. Note that $\deg g \leq d\cdot k\cdot s$ and g is computable by a $\Pi\Sigma\wedge$ circuit of size $O(s)$. Invoke Lemma 2.4 to obtain a hitting set $\mathcal{H}$, then evaluate g on every point of $\mathcal{H}$ to find an element $\boldsymbol{\alpha} \in \mathcal{H}$ such that $g(\boldsymbol{\alpha}) \neq 0$. We emphasize that in the whitebox setting all $U_{i,0}$ are readily available for evaluation. Since the size of the set is $\text{poly}(s)$ and each evaluation takes $\text{poly}(s)$ time, this preliminary step will add $\text{poly}(s)$ time to the overall time complexity. Moreover, we obtain the $\boldsymbol{\alpha} \in \mathbb{F}^n$ which possesses the required property.

To capture the non-zeroness, consider a 1-1 homomorphism $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[\boldsymbol{x}, z]$ such that $x_i \mapsto z \cdot x_i + a_i$ where a_i is the i-th coordinate of $\boldsymbol{\alpha}$, obtained earlier. Invertibility implies that $f_0 = 0 \iff \Phi(f_0) = 0$. Now we proceed with the recursive algorithm which first reduces the identity testing from top-fanin k to $k-1$. Note that $k = 1$ is trivial.

First Step: Efficient reduction from k to $k-1$

By assumption, $\sum_{i=1}^{k} T_{i,0} = f_0$ and $T_{k,0} \neq 0$. Apply Φ both sides, then divide and derive:

$$\begin{aligned}
\sum_{i\in[k]} T_{i,0} = f_0 &\iff \sum_{i\in[k]} \Phi(T_{i,0}) = \Phi(f_0) \\
&\iff \sum_{i\in[k-1]} \frac{\Phi(T_{i,0})}{\Phi(T_{k,0})} + 1 = \frac{\Phi(f_0)}{\Phi(T_{k,0})} \\
&\implies \sum_{i\in[k-1]} \partial_z\left(\frac{\Phi(T_{i,0})}{\Phi(T_{k,0})}\right) = \partial_z\left(\frac{\Phi(f_0)}{\Phi(T_{k,0})}\right) \\
&\iff \sum_{i=1}^{k-1} \frac{\Phi(T_{i,0})}{\Phi(T_{k,0})} \cdot \mathrm{dlog}_z\left(\frac{\Phi(T_{i,0})}{\Phi(T_{k,0})}\right) = \partial_z\left(\frac{\Phi(f_0)}{\Phi(T_{k,0})}\right). \qquad (3.1)
\end{aligned}$$

Here onwards we say dlog to mean dlog_z, unless stated otherwise. Define the following:

- $\mathsf{R}_1 := \mathbb{F}[z]/\langle z^d\rangle$. Note that, Equation (3.1) holds over $\mathsf{R}_1(\boldsymbol{x})$.
- $\widetilde{T}_{i,1} := \Phi(T_{i,0})/\Phi(T_{k,0}) \cdot \mathrm{dlog}(\Phi(T_{i,0})/\Phi(T_{k,0}))$, $\forall\, i \in [k-1]$.
- $f_1 := \partial_z(\Phi(f_0)/\Phi(T_{k,0}))$, over $\mathsf{R}_1(\boldsymbol{x})$.

Definability of $T_{i,1}$ and f_1. It is easy to see that these are well-defined terms. Here, we emphasize that we do not exactly compute/store $\widetilde{T}_{i,1}$ as a fraction where the degree in z is $< d$; instead it is computed as an element in $\mathbb{F}(z,\boldsymbol{x})$, where z is a formal variable. Formally, we compute $T_{i,1} \in \mathbb{F}(z,\boldsymbol{x})$, such that $\widetilde{T}_{i,1} = T_{i,1}$, over $\mathsf{R}_1(\boldsymbol{x})$. We keep track of the degree of z in $T_{i,1}$. Thus, $\sum_{i\in[k-1]} T_{i,1} = f_1$, over $\mathsf{R}_1(\boldsymbol{x})$.

The 'iff' condition. To show that our one step of DiDI has reduced to the identity testing of $\mathsf{Gen}(k-1,\cdot)$, we need a necessary and a sufficient condition. So far equality in Equation (3.1) over $\mathsf{R}_1(\boldsymbol{x})$ is *one-sided*. Note that $f_1 \neq 0$ implies $\mathrm{val}_z(f_1) < d =: d_1$. By assumption, $\Phi(T_{k,0})$ is invertible over $\mathsf{R}_1(\boldsymbol{x})$. Further, $f_1 = 0$, over $\mathsf{R}_1(\boldsymbol{x})$, which implies –

1. Either, $\Phi(f_0)/\Phi(T_{k,0})$ is z-free. Then $\Phi(f_0)/\Phi(T_{k,0}) \in \mathbb{F}(\boldsymbol{x})$, which further implies it is in $\mathbb{F}$, because of the map Φ (z-free implies $\boldsymbol{x}$-free, by substituting $z = 0$). Also, note that $f_0, T_{k,0} \neq 0$ implies $\Phi(f_0)/\Phi(T_{k,0})$ is a *nonzero* element in $\mathbb{F}$. Thus, it suffices to check whether $\Phi(f_0)|_{z=0}$ is non-zero or not.

2. Or, $\partial_z(\Phi(f_0)/\Phi(T_{k,0})) = z^{d_1} \cdot p$ where $p \in \mathbb{F}(z,\boldsymbol{x})$ such that $\mathrm{val}_z(p) \geq 0$. By a simple power series expansion, one can show that $p \in \mathbb{F}(x)[\![z]\!]$.

   Lemma 3.2 (Valuation). *Consider $f \in \mathbb{F}(\boldsymbol{x},y)$ such that $\mathrm{val}_y(f) \geq 0$. Then, $f \in \mathbb{F}(\boldsymbol{x})[\![y]\!] \bigcap \mathbb{F}(\boldsymbol{x},y)$.*

Proof sketch. Let $f = g/h$, where $g, h \in \mathbb{F}[\boldsymbol{x}, y]$. Now, $\mathrm{val}_y(f) \geq 0$, implies $\mathrm{val}_y(g) \geq \mathrm{val}_y(h)$. Let $\mathrm{val}_y(g) = d_1$ and $\mathrm{val}_y(h) = d_2$, where $d_1 \geq d_2 \geq 0$. Write $g = y^{d_1} \cdot \tilde{g}$ and $h = y^{d_2} \cdot \tilde{h}$. Write, $\tilde{h} = h_0 + h_1\, y + h_2\, y^2 + \ldots + h_d\, y^d$, for some d. Note that $h_0 \neq 0$. Thus,

$$\begin{aligned} f &= y^{d_1-d_2} \cdot \tilde{g}/(h_0 + h_1\, y + \ldots + h_d\, y^d) \\ &= y^{d_1-d_2} \cdot (\tilde{g}/h_0) \cdot (1 + (h_1/h_0)\, y + \ldots + (h_d/h_0)\, y^d)^{-1} \ \in \mathbb{F}(\boldsymbol{x})[\![y]\!] \,. \end{aligned}$$

The last conclusion follows by the inverse identity in the power-series ring. □

Hence, $\Phi(f_0)/\Phi(T_{k,0}) = z^{d_1+1} \cdot q$ where $q \in F(\boldsymbol{x})[\![z]\!]$, that is,

$$\Phi(f_0)/\Phi(T_{k,0}) \in \langle z^{d_1+1} \rangle_{\mathbb{F}(\boldsymbol{x})[\![z]\!]} \implies \mathrm{val}_z(\Phi(f_0)) \geq d+1,$$

a contradiction.

Conversely, it is obvious that $f_0 = 0$ implies $f_1 = 0$. Thus, we have proved the following

$$\sum_{i\in[k]} T_{i,0} \neq 0 \ \text{ over } \mathbb{F}[\boldsymbol{x}] \iff \sum_{i\in[k-1]} T_{i,1} \neq 0 \ \text{ over } \mathsf{R}_1(\boldsymbol{x}), \ \text{ or }, \ 0 \neq \Phi(f_0)|_{z=0} \in \mathbb{F} \,.$$

Eventually, we show that $T_{i,1} \in (\Pi\Sigma \wedge /\Pi\Sigma\wedge) \cdot (\Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge)$, over $\mathsf{R}_1(\boldsymbol{x})$, with polynomial blowup in size (Claim 3.4). So, the above circuit is in $\mathsf{Gen}(k-1, \cdot)$, over $\mathsf{R}_1(\boldsymbol{x})$, which we recurse on to finally give the identity testing. The subsequent steps will be a bit more tricky:

Induction step

Assume that we are in the j-th step ($j \geq 1$). Our induction hypothesis assumes –

1. $\sum_{i\in[k-j]} T_{i,j} = f_j$, over $\mathsf{R}_j(\boldsymbol{x})$, where $\mathsf{R}_j := \mathbb{F}[z]/\langle z^{d_j} \rangle$ for $d_j < d$, and $T_{i,j} \neq 0$.
2. $\mathrm{val}_z(T_{i,j}) \geq 0, \forall i \in [k-j]$.
3. Non-zero preserving iff condition

$$\begin{aligned} f \neq 0, \text{ over } \mathbb{F}[\boldsymbol{x}] \iff & f_j \neq 0, \text{ over } \mathsf{R}_j(\boldsymbol{x}), \\ & \text{or } \bigvee_{i=0}^{j-1} \left((f_i/T_{k-i,i})|_{z=0} \neq 0, \text{ over } \mathbb{F}(\boldsymbol{x}) \right) \end{aligned}$$

4. Here, $T_{i,j} =: \left(U_{i,j}/V_{i,j}\right) \cdot \left(P_{i,j}/Q_{i,j}\right)$, where $U_{i,j}, V_{i,j} \in \Pi\Sigma\wedge$, and $P_{i,j}, Q_{i,j} \in \Sigma\wedge\Sigma\wedge$, each in $\mathsf{R}_j[\boldsymbol{x}]$. Think of them being computed as $\mathbb{F}(z, \boldsymbol{x})$, with the degrees being tracked. Without loss of generality, assume that $\mathrm{val}_z(T_{k-j,j})$ is the minimal among all $T_{i,j}$'s.
5. $U_{i,j}\big|_{z=0}, V_{i,j}\big|_{z=0} \in \mathbb{F}\backslash\{0\}$.

We follow as before without applying homomorphism any further. Note that the 'or condition' in the Hypothesis 3 is similar to the $j = 0$ case except that there is no Φ: this is because $\Phi(f_0)|_{z=0} \neq 0 \iff \Phi(f_0/T_{k,0})|_{z=0} \neq 0$. This condition just separates the derivative from the constant-term.

Efficient reduction from $k-j$ to $k-j-1$. Let $\mathrm{val}_z(T_{i,j}) =: v_{i,j}$, for all $i \in [k-j]$. Note that

$$\min_i \mathrm{val}_z(T_{i,j}) = \min_i \mathrm{val}_z(P_{i,j}/Q_{i,j}) = v_{k-j,j}$$

since $\mathrm{val}_z(U_{i,j}) = \mathrm{val}_z(V_{i,j}) = 0$ (else we reorder). We remark that $0 \leq v_{i,j} < d_j$ for all i's in j-th step; upper-bound is strict, since otherwise $T_{i,j} = 0$ over $\mathsf{R}_j(x)$.

Similar to the first step, we divide with $T_{k-j,j}$ which has min val and then derive:

$$\begin{aligned}
\sum_{i\in[k-j]} T_{i,j} \;=\; f_j \iff & \sum_{i\in[k-j-1]} T_{i,j}/T_{k-j,j} \;+\; 1 \;=\; f_j/T_{k-j,j} \\
\implies & \sum_{i\in[k-j-1]} \partial_z(T_{i,j}/T_{k-j,j}) \;=\; \partial_z(f_j/T_{k-j,j}) \\
\iff & \sum_{i=1}^{k-j-1} T_{i,j}/T_{k-j,j} \cdot \mathrm{dlog}(T_{i,j}/T_{k-j,j}) \;=\; \partial_z(f_j/T_{k-j,j}) \qquad (3.2)
\end{aligned}$$

Define the following:

- $\mathsf{R}_{j+1} := \mathbb{F}[z]/\langle z^{d_{j+1}}\rangle$, where $d_{j+1} := d_j - v_{k-j,j} - 1$.
- $\widetilde{T}_{i,j+1} := T_{i,j}/T_{k-j,j} \cdot \mathrm{dlog}(T_{i,j}/T_{k-j,j})$, $\forall\, i \in [k-j-1]$.
- $f_{j+1} := \partial_z(f_j/T_{k-j,j})$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$.

We emphasize again that we do not exactly compute $\widetilde{T}_{i,j+1} \bmod z^{d_{j+1}}$; instead, it is computed as a fraction in $\mathbb{F}(z,\boldsymbol{x})$, with formal z. Formally, we compute $T_{i,j+1} \in \mathbb{F}(z,\boldsymbol{x})$, such that $\widetilde{T}_{i,j+1} = T_{i,j+1}$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$. We keep track of the degree of z in $T_{i,j+1}$. Next, we will show that all the inductive hypotheses assumed hold in the j^{th} step as well.

(Hypothesis 1): Definability of $T_{i,j+1}$ and f_{j+1}. By the minimal valuation assumption, it follows that $\mathrm{val}(f_j) \geq v_{k-j,j}$, and thus $\widetilde{T}_{i,j+1}$ and f_{j+1} are all well-defined over $\mathsf{R}_{j+1}(\boldsymbol{x})$. Note that, Equation (3.2) holds over $\mathsf{R}_{j+1}(\boldsymbol{x})$ as $d_{j+1} < d_j$ (because, whatever identity holds true mod z^{d_j} must hold mod $z^{d_{j+1}}$ as well). Hence, we must have $\sum_{i=1}^{k-j-1} \widetilde{T}_{i,j+1} = f_{j+1}$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$ thus proving the induction Hypothesis 1.

Hypothesis (2): Positivity of Valuation. Since we divide by the min val, by definition we immediately get $\mathrm{val}_z(T_{i,j+1}) \geq 0$ proving the hypothesis. Further, we claim that min val computation in DiDI is easy. For this, recall from the definition of valuation

$$\min_i \mathrm{val}_z(P_{i,j}/Q_{i,j}) = \min_i(\mathrm{val}_z(P_{i,j}) - \mathrm{val}_z(P_{i,j})).$$

Therefore, for min val we compute $\mathrm{val}_z(P_{i,j})$ and $\mathrm{val}_z(Q_{i,j})$ for all $i \in [k-j]$.

Here is an important lemma which shows that the coefficient of y^e of a polynomial $f(\boldsymbol{x}, y) \in \mathbb{F}[\boldsymbol{x}, y]$, computed by a $\Sigma\wedge\Sigma\wedge$ circuit, can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit.

Lemma 3.3 (Coefficient extraction)**.** *Let $f(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$ be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s and degree d. Then, $\mathrm{coef}_{y^e}(f) \in \mathbb{F}[\boldsymbol{x}]$ can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit of size $O(sd)$, over $\mathbb{F}[\boldsymbol{x}]$.*

Proof Sketch. Let, $f = \sum_i \alpha_i \cdot g_i^{e_i}$. Of course, $e_i \leq s$ and $\deg_y(f) \leq d$. Thus, write $f = \sum_{i=0}^{d} f_i \cdot y^i$, where $f_i \in \mathbb{F}[\boldsymbol{x}]$. We can interpolate on $d+1$-many distinct points $y \in \mathbb{F}$ and conclude that f_i has a $\Sigma\wedge\Sigma\wedge$ circuit of size at most $O(sd)$. □

Using Lemma 3.3 we know $\mathrm{coef}_{z^e}(P_{i,j})$ and $\mathrm{coef}_{z^e}(Q_{i,j})$ are in $\Sigma\wedge\Sigma\wedge$ over $F[\boldsymbol{x}]$. We can keep track of z degree and thus interpolate to find the minimum $e < d_j$ such that the computed coefficients are $\neq 0$, which gives the respective val.

Hypothesis (3): The 'iff' condition. The above Equation (3.2) allows us to reduce from $k-j$-summands to $k-j-1$. But we want a necessary and sufficient condition to efficiently reduce the identity testing. If $f_{j+1} \neq 0$, then $\mathrm{val}_z(f_{j+1}) < d_{j+1}$. Further, $f_{j+1} = 0$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$ implies–

1. Either, $f_j/T_{k-j,j}$ is z-free. This implies it is in $\mathbb{F}(\boldsymbol{x})$. Now, if indeed $f_0 \neq 0$, then the computed $T_{i,j}$ as well as f_j must be non-zero over $\mathbb{F}(z, \boldsymbol{x})$, by the induction hypothesis (as they are non-zero over $\mathsf{R}_j(\boldsymbol{x})$). However,

$$\left(\frac{T_{i,j}}{T_{k-j,j}}\right)\Big|_{z=0} = \left(\frac{U_{i,j}\cdot V_{k-j,j}}{U_{k-j,j}\cdot V_{i,j}}\right)\Big|_{z=0} \cdot \left(\frac{P_{i,j}\cdot Q_{k-j,j}}{P_{k-j,j}\cdot Q_{i,j}}\right)\Big|_{z=0}$$
$$\in \mathbb{F}\cdot\left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right).$$

   Thus,

$$\frac{f_j}{T_{k-j,j}} \in \sum \mathbb{F}\cdot\left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right) \in \left(\frac{\Sigma\wedge\Sigma\wedge}{\Sigma\wedge\Sigma\wedge}\right).$$

   Here we crucially use that $\Sigma\wedge\Sigma\wedge$ is closed under multiplication (Lemma 2.11). Thus, this identity testing can be done in poly-time (Lemma 2.9). For detailed time-complexity and calculations, see Claim 3.4 and its subsequent paragraph.

2. Or, $\partial_z(f_j/T_{k-j,j}) = z^{d_{j+1}}\cdot p$, where $p \in \mathbb{F}(z, \boldsymbol{x})$ such that $\mathrm{val}_z(p) \geq 0$. By a simple power series expansion, one concludes that $p \in \mathbb{F}(\boldsymbol{x})[\![z]\!]$ (Lemma 3.2). Hence, one concludes that

$$\frac{f_j}{T_{k-j,j}} \in \left\langle z^{d_{j+1}+1}\right\rangle_{\mathbb{F}(\boldsymbol{x})[\![z]\!]} \implies \mathrm{val}_z(f_j) \geq d_j,$$

   that is, $f_j = 0$, over $\mathsf{R}_j(\boldsymbol{x})$.

Conversely, $f_j = 0$, over $\mathsf{R}_j(\boldsymbol{x})$, implies

$$\begin{aligned}\mathrm{val}_z(f_j) \geq d_j &\implies \mathrm{val}_z\left(\partial_z\left(\frac{f_j}{T_{k-j,j}}\right)\right) \geq d_j - v_{k-j,j} - 1\\ &\implies f_{j+1} = 0, \text{ over } \mathsf{R}_{j+1}(\boldsymbol{x}).\end{aligned}$$

Thus, we have proved that $\sum_{i\in[k-j]} T_{i,j} \neq 0$ over $\mathsf{R}_j(\boldsymbol{x})$ iff

$$\sum_{i\in[k-j-1]} T_{i,j+1} \neq 0 \text{ over } \mathsf{R}_{j+1}(\boldsymbol{x}) \text{ , or , } \quad 0 \neq \left(\frac{f_j}{T_{k-j,j}}\right)\Big|_{z=0} \in \mathbb{F}(\boldsymbol{x}) \text{ .}$$

Therefore induction hypothesis (3) holds.

Hypothesis (4): Size analysis. We will show that $T_{i,j+1} \in (\Pi\Sigma \wedge /\Pi\Sigma\wedge) \cdot (\Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge)$, over $\mathsf{R}_{j+1}(\boldsymbol{x})$, with only polynomial blowup in size. Let $\mathrm{size}(T_{i,j}) \leq s_j$, for $i \in [k-j]$, and $j \in [k]$. Note that, by assumption, $s_0 \leq s$.

Claim 3.4 (Final size). *$T_{1,k-1} \in (\Pi\Sigma \wedge /\Pi\Sigma\wedge) \cdot (\Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge)$ of size $s^{O(k7^k)}$, over $\mathsf{R}_{k-1}(\boldsymbol{x})$.*

Proof. Steps $j = 0$ and $j > 0$ are slightly different because of the Φ. However the main idea of using power-series is the same which eventually shows that $\mathrm{dlog}(\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge$.

We first deal with $j = 0$. Let $A - z \cdot B = \Phi(g) \in \Sigma\wedge$, for some $A \in \mathbb{F}$ and $B \in \mathsf{R}_1[\boldsymbol{x}]$. Note that $A \neq 0$ because of the map Ψ. Further, $\mathrm{size}(B) \leq O(d \cdot \mathrm{size}(g))$, as a single monomial of the form x^e can produce $d+1$-many monomials. Over $\mathsf{R}_1(\boldsymbol{x})$,

$$\mathrm{dlog}(\Phi(g)) = -\frac{\partial_z(B\cdot z)}{A(1-\frac{B}{A}\cdot z)} = -\frac{\partial_z(B\cdot z)}{A} \cdot \sum_{i=0}^{d_1-1}\left(\frac{B}{A}\right)^i \cdot z^i \text{ .} \tag{3.3}$$

B^i has a trivial $\wedge\Sigma\wedge$-circuit of size $O(d \cdot \mathrm{size}(g))$. Also, $\partial_z(B \cdot z)$ has a $\Sigma\wedge$-circuit of size at most $O(d \cdot \mathrm{size}(g))$. Using waring identity (Lemma 2.10), we get that each $\partial_z(B \cdot z) \cdot (B/A)^i \cdot z^i$ has size $O(i \cdot d \cdot \mathrm{size}(g))$, over $\mathsf{R}_1(\boldsymbol{x})$. Summing over $i \in [d_1 - 1]$, the overall size is at most $O(d_1^2 \cdot d \cdot \mathrm{size}(g)) = O(d^3 \cdot \mathrm{size}(g))$, as $d_0 = d_1 = d$.

For the j-th step, we emphasize that the degree could be larger than d. Assume that syntactic degree of denominator and numerator of $T_{i,j}$ (each in $\mathbb{F}[\boldsymbol{x}, z]$) are bounded by D_j (it is *not* d_j as seen above; this is to save on the trouble of mod-computation at each step). Of course, $D_0 < d \leq s$.

For $j > 0$, the above summation in Equation (3.3) is over $\mathsf{R}_j(\boldsymbol{x})$. However the degree could be D_j (possibly more than d_j) of the corresponding A and B. Thus, the overall size after the power-series expansion would be $O(D_j^2 \cdot d \cdot \mathrm{size}(g))$.

Using Lemma 3.5, we can show that $\mathrm{dlog}(P_{i,j}) \in \Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge$ (similarly for $Q_{i,j}$), of size $O(D_j^2 \cdot s_j)$. Also $\mathrm{dlog}(U_{i,j} \cdot V_{k-j,j}) \in \sum \mathrm{dlog}(\Sigma\wedge)$, that is, sum of action of dlog on $\Sigma\wedge$ (since dlog linearizes product); and it can be computed by the above formulation.

Thus, $\mathrm{dlog}(T_{i,j}/T_{k-j,j})$ is a sum of 4 $\Sigma\wedge\Sigma\wedge /\Sigma\wedge\Sigma\wedge$ circuit of size at most $O(D_j^2\, s_j)$ and a $\Sigma\wedge\Sigma\wedge$ circuit of size $O(D_j^2 d_j s_j)$ (from the above power-series computation) [Note: that we summed

up the $\Sigma\wedge\Sigma\wedge$-expressions from dlog$(\Sigma\wedge)$ together]. Additionally the syntactic degree of each denominator and numerator (of the $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$) is $O(D_j)$. We rewrite the 4 expressions (each of $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$) and express it as a single $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ using the waring identity (Lemma 2.11), with the size blowup of $O(D_j^{12}\, s_j^4)$; here the syntactic degree blows up to $O(D_j)$. Finally we add the remaining $\Sigma\wedge\Sigma\wedge$ circuit (of size $O(D_j^3 s_j)$ and degree $O(dD_j)$) to get $O(s_j^5 D_j^{16} d)$. To bound this, we need to understand the degree bound D_j.

Finally we need to multiply $T_{i,j}/T_{k-j,j} \in (\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ where each $\Sigma\wedge\Sigma\wedge$ is a product of two $\Sigma\wedge\Sigma\wedge$ expression of size s_j and syntactic degree D_j; clubbed together causing a blowup of $O(D_j \cdot s_j^2)$. Hence multiplying it with $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ expression obtained from dlog computation above gives size blowup of $s_{j+1} = s^7 \cdot D_j^{O(1)} \cdot d$.

Computing $T_{i,j}/T_{k-j,j}$ increases the syntactic degree 'slowly'; which is much less than the size blowup. As mentioned before, the deg-blowup in the dlog-computation is $O(dD_j)$ and in the clearing of the four expressions, it is just $O(D_j)$. Thus, $D_{j+1} = O(dD_j) \implies D_j = d^{O(j)}$.

The recursion on the size is $s_{j+1} = s_j^7 \cdot d^{O(j)}$. Using $d \le s$ we deduce, $s_j = (sd)^{O(j\cdot 7^j)}$. In particular, s_{k-1}, size after $k-1$ steps is $s^{O(k\cdot 7^k)}$. This computation quantitatively establishes induction hypothesis (4). □

Hypothesis (5): Invertibility of $\Pi\Sigma\wedge$-circuits. For invertibility, we want to emphasise that the dlog computation plays a crucial role here. In the following lemma we claim that the action dlog$(\Sigma\wedge\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$, is of poly-size.

Lemma 3.5 (Differentiation)**.** *Let $f(\boldsymbol{x}, y) \in \mathbb{F}[y][\boldsymbol{x}]$ be computed by a $\Sigma\wedge\Sigma\wedge$ circuit of size s and degree d. Then, $\partial_y(f)$ can be computed by a small $\Sigma\wedge\Sigma\wedge$ circuit of size $O(sd^2)$, over $\mathbb{F}[y][\boldsymbol{x}]$.*

Proof Sketch. Lemma 3.3 shows that each f_e has $O(sd)$ size circuit where $f = \sum_e f_e\, y^e$. Doing this for each $e \in [0, d]$ gives a blowup of $O(sd^2)$. □

Similarly consider the action on $\Pi\Sigma\wedge$. We know dlog distributes the product additively, so it suffices to work with dlog$(\Sigma\wedge)$; and earlier in Claim 3.4 we saw that dlog$(\Sigma\wedge) \in \Sigma\wedge\Sigma\wedge$ of poly-size. Assuming these, we simplify

$$\frac{T_{i,j}}{T_{k-j,j}} = \frac{U_{i,j}\cdot V_{k-j,j}}{V_{i,j}\cdot U_{k-j,j}} \cdot \frac{P_{i,j}\cdot Q_{k-j,j}}{Q_{i,j}\cdot P_{k-j,j}},$$

and its dlog. Thus, using Equation (3.2), $U_{i,(j+1)}$ grows to $U_{i,j}\cdot V_{k-j,j}$ (and similarly $V_{i,(j+1)}$). This also means: $U_{i,(j+1)}\big|_{z=0} \in \mathbb{F}\setminus\{0\}$ and thereby proving the hypothesis.

Final time complexity

The above proof actually shows that $T_{1,k-1}$ is in $\mathsf{Gen}(1, s^{O(k\cdot 7^k)})$ over $\mathsf{R}_{k-1}(\boldsymbol{x})$; and that the degree bound on z (over $\mathbb{F}[z, \boldsymbol{x}]$, keeping denominator and numerator 'in place') is $D_{k-1} = d^{O(k)}$. We cannot directly use the identity testing algorithms of the constituent simpler models due to $\mathsf{R}_{k-1}(\boldsymbol{x})$. Moreover, using hypothesis (2) and Lemma 3.2 we know that $T_{1,k-1} \in \mathbb{F}(\boldsymbol{x})[\![z]\!]$ and

it suffices to test the identity of the first term of the power series: $T_{1,k-1}|_{z=0}$ over $\mathbb{F}(\boldsymbol{x})$. Note that, hypothesis (5) guarantees that $\Pi\Sigma\wedge$ part remains non-zero at $z=0$ evaluation. However, $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ may be undefined. For this, we keep track of z degree of numerator and denominator, which will be polynomially bounded as seen in the discussion above. We can easily interpolate and cancel the z power to make it work. Basically this shows that to test $T_{1,k-1}$ we need to test $z^e\cdot\Sigma\wedge\Sigma\wedge$ over $\mathbb{F}[\boldsymbol{x}]$ where $e\geq 0$ due to positive valuation. Whitebox PIT of $\Sigma\wedge\Sigma\wedge$ is in poly-time using Lemma 2.9, and testing z^e is possible using Lemma 2.1 with appropriate degree bound. The proof above is constructive: we calculate $U_{i,j+1}$ (and other terms) from $U_{i,j}$ explicitly. Gluing everything together we conclude this part can be done in $s^{O(k7^k)}$ time.

What remains is to test the $z=0$-part of induction hypothesis (3); it could *short-circuit* the recursion much before $j=k-1$. As we mentioned before, in this case, we need to test the identity only of a $\Sigma\wedge\Sigma\wedge$ circuit. At the j-th step, when we substitute $z=0$, the size of each $T_{i,j}$ can be at most s_j (by definition). We need to do PIT on a simpler model: $\sum^{[k-j]}\mathbb{F}\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$. We can clear out and express this as a single $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ expression; with a size blowup of $s_j^{O(k-j)}\leq(sd)^{O(j(k-j)7^j)}$. Since this case could short-circuit the recursion, to bound the final time complexity, we need to consider the j which maximises the exponent.

Lemma 3.6. *Let $k\in\mathbb{N}$, and $h(x):=x(k-x)7^x$. Then, $\max_{i\in[k-1]}h(i)=h(k-1)$.*

Proof Sketch. Differentiate to get $h'(x)=(k-x)7^x-x7^x+x(k-x)(\log 7)7^x=7^x\cdot[x^2(-\log 7)+x(k\log 7-2)+k]$. It vanishes at

$$x=\left(\frac{k}{2}-\frac{1}{\log 7}\right)+\sqrt{\left(\frac{k}{2}-\frac{1}{\log 7}\right)^2-\frac{k}{\log 7}}\,.$$

Thus, h is maximised at the integer $x=k-1$. □

Therefore, $\max_{j\in[k-1]}j(k-j)7^j=(k-1)7^{k-1}$. Finally, use Lemma 2.9 for the base-case whitebox PIT. Thus, the final time complexity is $s^{O(k\cdot 7^k)}$.

Here we also remark that when $z=0$, the $\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge$ expression may be undefined. However, we keep track of z degree of numerator and denominator, which will be polynomially bounded as seen in the discussion above. We can easily interpolate and cancel the z power to make it work.

Bit complexity. It is routine to show that the bit-complexity is really what we claim. Initially, the given circuit has bit-complexity s. The main blowup happens due to the dlog-computation which is a poly-size blowup. We also remark that while using Lemma 2.11 (using Lemma 2.10), we *may* need to go to a field extension of at most $s^{O(k)}$ (because of the $\varepsilon(i)$ and correspondingly the constants $\gamma_{\varepsilon(2),\ldots,\varepsilon(k)}$, but they still are $s^{O(k)}$-bits). Also, Theorem 2.2 and Lemma 2.9 computations blowup bit-complexity polynomially. This concludes the proof. □

Remark 3.7. 1. The above method does *not* give whitebox PIT (in poly-time) for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$, as we do not know poly-time whitebox PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$. However, the above methods do show that whitebox-PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ polynomially *reduces* to whitebox-PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$.

2. DiDI-technique can be used to give whitebox PIT for the general bloated model $\mathsf{Gen}(k, s)$.
3. The above proof works when the characteristic is $\geq d$. This is because the nonzeroness remains *preserved* after derivation wrt z.

3.1 Algorithm

The whitebox PIT for Theorem 1.1, that is discussed in Section 3, appears (below) as Algorithm 1.

Algorithm 1 Whitebox PIT Algorithm for $\Sigma^{[k]}\Pi\Sigma\wedge$-circuits

INPUT: $f = T_1 + \ldots + T_k \in \Sigma^{[k]}\Pi\Sigma\wedge$**, a whitebox circuit of size** s **over** $\mathbb{F}[\boldsymbol{x}]$
OUTPUT: 0**, if** $f \equiv 0$**, and** 1**, if non-zero.**

1: Let $\Psi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z]$, be a sparse-PIT map, using [57] (Theorem 2.2). Apply it on f and check whether $\Psi(f) \stackrel{?}{=} 0$. If non-zero, `output` 1
2: Obtain a point $\boldsymbol{\alpha} = (a_1, \ldots, a_n) \in \mathbb{F}^n$ from Hitting Set $\mathcal{H}$ of $\Pi\Sigma\wedge$ such that $T_i|_{\boldsymbol{x}=\boldsymbol{\alpha}} \neq 0$, for all $i \in [k]$. And define $\Phi : x_i \mapsto z \cdot x_i + a_i$. Check $\sum_{i\in[k-1]} \partial_z(\Phi(T_i)/\Phi(T_k)) \stackrel{?}{=} 0 \bmod z^{d_1}$ $(d_1 := s)$ as follows:
3: Consider each $T_{i,1} := \partial_z(\Phi(T_i)/\Phi(T_k))$ over $R_1(\boldsymbol{x})$, where $R_1 := \mathbb{F}[z]/\langle z^{d_1}\rangle$. Use dlog computation (Claim 3.4), to write each $T_{i,1}$ in a 'bloated' form as $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$.
4: **for** $j \leftarrow 1$ **to** $k-1$ **do**
5: Reduce the top-fanin at each step using '**Di**vide and **D**erive' technique. Assume that at j-th step, we have to check the identity: $\sum_{i\in[k-j]} T_{i,j} \stackrel{?}{=} 0$ over $R_j(\boldsymbol{x})$, where $R_j := \mathbb{F}[z]/\langle z^{d_j}\rangle$, each $T_{i,j}$ has a $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ representation and therein each $\Pi\Sigma\wedge|_{z=0} \in \mathbb{F}\setminus\{0\}$.
6: Compute $v_{k-j,j} := \min_i \mathrm{val}_z(T_{i,j})$; by reordering it is for $i = k-j$. To compute $v_{k-j,j}$, use coefficient extraction (Lemma 3.3) and $\Sigma\wedge\Sigma\wedge$-circuit PIT (Lemma 2.9).
7: '**D**ivide' by $T_{k-j,j}$ and check whether $\left(\sum_{i\in[k-j-1]}(T_{i,j}/T_{k-j,j})+1\right)\Big|_{z=0} \stackrel{?}{=} 0$. Note: this expression is in $(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$. Use— (1) $\Pi\Sigma\wedge|_{z=0} \in \mathbb{F}$, and (2) *closure* of $\Sigma\wedge\Sigma\wedge$ under multiplication. Finally, do PIT on this by Lemma 2.9.
8: If it is non-zero, `output` 1, otherwise '**D**erive' wrt z and '**I**nduct' on $\left(\sum_{i\in[k-j-1]}\partial_z(T_{i,j}/T_{k-j,j})\right) \stackrel{?}{=} 0$, over $R_{j+1}(\boldsymbol{x})$ where $R_{j+1} := \mathbb{F}[z]/\langle z^{d_j - v_{k-j,j}-1}\rangle$.
9: Again using dlog (Claim 3.4), show that $T_{i,j+1} := \partial_z(T_{i,j}/T_{k-j,j})$ has small $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$-circuit over $R_{j+1}(\boldsymbol{x})$. So call the algorithm on $\sum_{i\in[k-j-1]} T_{i,j+1} \stackrel{?}{=} 0$.
10: $j \leftarrow j+1$.
11: **end for**
12: At the end, $j = k-1$. Perform PIT (Lemma 2.9) on the single $(\Pi\Sigma\wedge/\Pi\Sigma\wedge)\cdot(\Sigma\wedge\Sigma\wedge/\Sigma\wedge\Sigma\wedge)$ circuit, over $R_{k-1}(\boldsymbol{x})$. If it is zero, `output` 0 otherwise `output` 1.

Words of caution: Throughout the algorithm there are intermediate expressions to be stored

compactly. Think of them as 'special' circuits in $\boldsymbol{x}$, but over the *function-field* $\mathbb{F}(\boldsymbol{z})$. Keep track of their degrees wrt z; and that of the sizes of their fractions represented in 'bloated' circuit form.

4 Blackbox PIT

We will give the proof of Theorem 1.3 in this section. Before the details, we will state a few important definitions and lemmas from [5] to be referenced later.

Definition 4.1 (Transcendence Degree)**.** Polynomials $T_1, \ldots, T_m$ are called *algebraically dependent* if there exists a nonzero *annihilator* A such that $A(T_1, \ldots, T_m) = 0$. *Transcendence degree* is the size of the largest subset $S \subseteq \{T_1, \ldots, T_m\}$ that is algebraically independent. Then S is called a *transcendence basis*.

Definition 4.2 (Faithful homomorphism)**.** A homomorphism $\Phi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{y}]$ is faithful for $\boldsymbol{T}$ if $\mathrm{trdeg}_{\mathbb{F}}(\boldsymbol{T}) = \mathrm{trdeg}_{\mathbb{F}}(\Phi(\boldsymbol{T}))$.

The reason for interest in faithful maps is due to its usefulness in preserving the identity as shown in the following fact.

Fact 4.3 (Theorem 2.4 [5])**.** *For any* $C \in \mathbb{F}[y_1, \ldots, y_m]$, $C(\boldsymbol{T}) = 0 \iff C(\Phi(\boldsymbol{T})) = 0$.

Here is an important criterion about the jacobian matrix which basically shows that it *preserves* algebraic independence.

Fact 4.4 (Jacobian criterion)**.** *Let* $\boldsymbol{f} \subset \mathbb{F}[\boldsymbol{x}]$ *be a finite set of polynomials of degree at most* d*, and* $\mathrm{trdeg}_{\mathbb{F}}(\boldsymbol{f}) \leq r$*. If char*$(\mathbb{F}) = 0$*, or char*$(\mathbb{F}) > d^r$*, then* $\mathrm{trdeg}_{\mathbb{F}}(\boldsymbol{f}) = \mathrm{rk}_{\mathbb{F}(x)} \mathcal{J}_{\boldsymbol{x}}(\boldsymbol{f})$.

The Jacobian criterion together with faithful maps gives a recipe to design a map which drastically reduces the number of variables, if trdeg is small.

Lemma 4.5 (Lemma 2.7 [5])**.** *Let* $\boldsymbol{T} \in \mathbb{F}[\boldsymbol{x}]$ *be a finite set of polynomials of degree at most* d *and* $\mathrm{trdeg}_{\mathbb{F}}(\boldsymbol{T}) \leq r$*, and char(F)=0 or* $> d^r$*. Let* $\Psi' : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[\boldsymbol{z}]$ *such that* $\mathrm{rk}_{\mathbb{F}(x)} \mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}) = \mathrm{rk}_{\mathbb{F}(z)} \Psi'(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$.

Then, the map $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z, t, \boldsymbol{y}]$*, such that* $x_i \mapsto (\sum_{j \in [r]} y_j t^{ij}) + \Psi'(x_i)$*, is a faithful homomorphism for* $\boldsymbol{T}$.

In the next section we will use these tools to prove Theorem 1.3(b). The proof and calculations for Theorem 1.3(a) are very similar.

4.1 PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$

We solve the PIT for a more general model than $\Sigma^{[k]}\Pi\Sigma\Pi$ by solving the following problem.

Problem 4.6. Let $\{T_i \mid i \in [m]\}$ be $\Pi\Sigma\Pi^{[\delta]}$ circuits of (syntactic) degree at most d and size s. Let the transcendence degree of T_i's, $\mathrm{trdeg}_{\mathbb{F}}(T_1, \ldots, T_m) = k \ll s$. Further, $C(x_1, \ldots, x_m)$ be a circuit of (size + deg) $< s'$. Design a blackbox-PIT algorithm for $C(T_1, \ldots, T_m)$.

Trivially, $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ is a very special case of the above setting. Let $\boldsymbol{T} := \{T_1, \ldots, T_m\}$. Let $\boldsymbol{T}_k := \{T_1, \ldots, T_k\}$ be a transcendence basis. For $T_i = \prod_j g_{ij}$, we denote the set $L(T_i) := \{g_{ij} \mid j\}$.

We want to find an explicit homomorphism $\Psi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}, z]$ such that $\Psi(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$ is of a 'nice' form. In the image we fix $\boldsymbol{x}$ suitably, to get a composed map $\Psi' : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z]$ such that $\mathrm{rk}_{\mathbb{F}(\boldsymbol{x})}\, \mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}) = \mathrm{rk}_{\mathbb{F}(z)}\, \Psi'(\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T}))$. Then, we can extend this map to $\Phi : \mathbb{F}[\boldsymbol{x}] \longrightarrow \mathbb{F}[z, \boldsymbol{y}, t]$ such that $x_i \mapsto (\sum_{j=1}^{k} y_j t^{ij}) + \Psi'(x_i)$, which is *faithful* Lemma 4.5. We show that the map Φ can be efficiently constructed using a scaling and shifting map (Ψ) which is eventually fixed by the hitting set (H' defining Ψ') of a $\Sigma \wedge \Sigma\Pi^{[\delta]}$ circuit. Overall, $\Phi(f)$ is a $k+2$-variate polynomial for which a trivial hitting set exists.

Without loss of generality, $\mathcal{J}_{\boldsymbol{x}}(\boldsymbol{T})$ is full rank with respect to the variable set $\boldsymbol{x}_k = (x_1, \ldots, x_k)$. Thus, by assumption, $J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) \neq 0$ (for notation, see Section 2). We want to construct a Ψ such that $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))$ has an 'easier' PIT. We have the following identity [5, Eqn. 3.1], from the linearity of the determinant, and the simple observation that $\partial_x(T_i) = T_i \cdot \left(\sum_j \partial_x(g_{ij})/g_{ij}\right)$, where $T_i = \prod_j g_{ij}$:

$$J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) = \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \left(\frac{T_1 \ldots T_k}{g_1 \ldots g_k}\right) \cdot J_{\boldsymbol{x}_k}(g_1, \ldots, g_k)\,. \tag{4.1}$$

The homomorphism Ψ. To ensure the invertibility of all $g \in \bigcup_i L(T_i)$ we proceed as in Section 3. Consider

$$h := \prod_{i \in [k]} \prod_{g \in L(T_i)} g = \prod_{i \in [\ell]} g,$$

where $g \in \bigcup_i L(T_i)$ and $\ell \leq k \cdot s$. Note that $\deg h \leq d \cdot k \cdot s$ and h is computable by $\Pi\Sigma\Pi$ circuit of size $O(s)$. Lemma 2.4 gives the relevant hitting set $\mathcal{H} \subseteq \mathbb{F}^n$ which contains an evaluation point $\boldsymbol{\alpha} = (a_1, \ldots, a_n)$ such that $h(\boldsymbol{\alpha}) \neq 0$ implying $g(\boldsymbol{\alpha}) \neq 0$, for all $g \in \bigcup_i L(T_i)$. We emphasise that, unlike the previous case, here in the blackbox setting, we *do not* have individual access to g to verify for the correct $\boldsymbol{\alpha}$. Thus, we try out all $\boldsymbol{\alpha} \in \mathcal{H}$ to see whichever works. If the input polynomial f is non-zero, then one such $\boldsymbol{\alpha}$ must exist. This search adds a multiplicative blowup of $\mathrm{poly}(s)$, since the size of $\mathcal{H}$ is $\mathrm{poly}(s)$.

Fix an $\boldsymbol{\alpha} = (a_1, \cdots, a_n) \in \mathcal{H}$ and define $\Psi : \mathbb{F}[\boldsymbol{x}] \to \mathbb{F}[\boldsymbol{x}, z]$ as $x_i \mapsto z \cdot x_i + a_i$. Denote the ring $\mathsf{R}[\boldsymbol{x}]$ where $\mathsf{R} := \mathbb{F}[z]/\langle z^D \rangle$, and $D := k \cdot (d-1) + 1$. Being 1-1, Ψ is clearly a non-zero preserving map. Moreover,

Claim 4.7. *$J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) = 0 \iff \Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) = 0$, over $\mathsf{R}[\boldsymbol{x}]$.*

Proof. As $\deg(T_i) \leq d$, each entry of the matrix can be of degree at most $d-1$; therefore $\deg(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) \leq k(d-1) = D-1$. Thus, $\deg_z(\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))) < D$. Hence, the conclusion. □

Equation (4.1) implies that

$$\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) = \Psi(T_1 \cdots T_k) \cdot \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \frac{\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))}{\Psi(g_1 \ldots g_k)}\,. \tag{4.2}$$

As T_i has product fanin s, the top-fanin in the sum in Equation (4.2) can be at most s^k. Then define,

$$\widetilde{F} := \sum_{g_1 \in L(T_1), \ldots, g_k \in L(T_k)} \frac{\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))}{\Psi(g_1 \ldots g_k)}, \quad \text{over } \mathsf{R}[\boldsymbol{x}]. \tag{4.3}$$

Well-definability of $\widetilde{F}$. Note that,

$$\Psi(g_i) \bmod z \neq 0 \implies 1/\Psi(g_1 \cdots g_k) \in \mathbb{F}[\![\boldsymbol{x}, z]\!].$$

Thus, RHS is an element in $\mathbb{F}[\![\boldsymbol{x}, z]\!]$ and taking mod z^D it is in $\mathsf{R}[\boldsymbol{x}]$. We remark that instead of minimally reducing mod z^D, we will work with an $F \in \mathbb{F}[z, \boldsymbol{x}]$ such that $F = \tilde{F}$ over $\mathsf{R}[\boldsymbol{x}]$. Further, we ensure that the degree of z is polynomially bounded.

Claim 4.8. *Over* $\mathsf{R}[\boldsymbol{x}]$, $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k)) = 0 \iff F = 0$.

Proof sketch. This follows from the invertibility of $\Psi(T_1 \cdots T_k)$ in $R[\boldsymbol{x}]$. □

The hitting set H'. By $J_{\boldsymbol{x}_k}(\boldsymbol{T}_k) \neq 0$, and Claim 4.7–4.8, we have $F \neq 0$ over $\mathsf{R}[\boldsymbol{x}]$. We want to find $H' \subseteq \mathbb{F}^n$, such that $\Psi(J_{\boldsymbol{x}_k}(\boldsymbol{T}_k))\big|_{\boldsymbol{x}=\boldsymbol{\alpha}} \neq 0$, for some $\boldsymbol{\alpha} \in H'$ (which will ensure the rank-preservation). Towards this, we will show (below) that F has $s^{O(\delta k)}$-size $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit over $\mathsf{R}[\boldsymbol{x}]$. Next, Theorem 2.8 provides the hitting set H' in time $s^{O(\delta^2 k \log s)}$.

Claim 4.9 (Main size bound). $F \in \mathsf{R}[\boldsymbol{x}]$ *has* $\Sigma \wedge \Sigma\Pi^{[\delta]}$*-circuit of size* $(s3^\delta)^{O(k)}$.

The proof studies the two parts of Equation (4.3)—

1. The numerator $\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k))$ has $O(3^\delta 2^k k! ks)$-size $\Sigma \wedge \Sigma\Pi^{[\delta-1]}$-circuit (see Lemma 4.12), and
2. $1/\Psi(g_1 \cdots g_k)$, for $g_i \in L(T_i)$ has $(s3^\delta)^{O(k)}$-size $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit; both over $\mathsf{R}[\boldsymbol{x}]$ (see Lemma 4.13).

We need the following two claims to prove the numerator size bound.

Claim 4.10. *Let* $g_i \in L(T_i)$*, where* $T_i \in \Pi\Sigma\Pi^{[\delta]}$ *of size at most* s*, then the polynomial* $J_{\boldsymbol{x}_k}(g_1, \ldots, g_k)$ *is computable by* $\Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$ *of size* $O(k!\, ks)$.

Proof Sketch. Each entry of the matrix has degree at most $\delta - 1$. Trivial expansion gives $k!$ top-fanin where each product (of fanin k) has size $\sum_i \text{size}(g_i)$. As, $\text{size}(T_i) \leq s$, trivially each $\text{size}(g_i) \leq s$. Therefore, the total size is $k! \cdot \sum_i \text{size}(g_i) = O(k!\, ks)$. □

Claim 4.11. *Let* $g \in \Sigma\Pi^\delta$*, then* $\Psi(g) \in \Sigma\Pi^\delta$ *of size* $3^\delta \cdot \text{size}(g)$ *(for* $n \gg \delta$*).*

Proof Sketch. Each monomial $\boldsymbol{x}^{\boldsymbol{e}}$ of degree δ, can produce $\prod_i (e_i+1) \leq ((\sum_i e_i + n)/n)^n \leq (\delta/n+1)^n$-many monomials, by AM-GM inequality as $\sum_i e_i \leq \delta$. As $\delta/n \to 0$, we have $(1 + \delta/n)^n \to e^\delta$. As $e < 3$, the upper bound follows. □

Lemma 4.12 (Numerator size). *$\Psi(J_{\boldsymbol{x}_k}(g_1,\dots,g_k))$ is computable by $\Sigma\wedge\Sigma\Pi^{[\delta-1]}$ of size $O(3^\delta\, 2^k k\, k!s) =: s_2$.*

Proof. In Claim 4.10 we showed that $J_{\boldsymbol{x}_k}(g_1,\dots,g_k) \in \Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$ of size $O(k!ks)$. Moreover, for a $g \in \Sigma\Pi^{[\delta-1]}$, we have $\Psi(g) \in \Sigma\Pi^{[\delta-1]}$ of size at most $3^\delta \cdot \text{size}(g)$, over $\mathsf{R}[\boldsymbol{x}]$ due to Claim 4.11).

Combining these, one concludes that $\Psi(J_{\boldsymbol{x}_k}(g_1,\dots,g_k)) \in \Sigma^{[k!]}\Pi^{[k]}\Sigma\Pi^{[\delta-1]}$, of size $O(3^\delta\, k!ks)$. We *convert* the Π-gate to $\wedge$ gate using waring identity (Lemma 2.10) which blows up the size by a multiple of 2^{k-1}. Thus, $\Psi(J_{\boldsymbol{x}_k}(g_1,\dots,g_k)) \in \Sigma\wedge\Sigma\Pi^{[\delta-1]}$ of size $O(3^\delta\, 2^k k\, k!s)$. □

In the following lemma, using power series expansion of expressions like $1/(1-a\cdot z)$, we conclude that $1/\Psi(g)$ has a small $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit, which would further imply the same for $1/\Psi(g_1\cdots g_k)$.

Lemma 4.13 (Denominator size). *Let $g_i \in L(T_i)$. Then, $1/\Psi(g_1\cdots g_k)$ can be computed by a $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size $s_1 := (s3^\delta)^{O(k)}$, over $\mathsf{R}[\boldsymbol{x}]$.*

Proof. Let $g \in L(T_i)$ for some i. Assume, $\Psi(g) = A - z\cdot B$, for some $A \in \mathbb{F}$ and $B \in \mathsf{R}[\boldsymbol{x}]$ of degree δ, with $\text{size}(B) \le 3^\delta \cdot s$, from Claim 4.11. Note that, over $\mathsf{R}[\boldsymbol{x}]$,

$$\frac{1}{\Psi(g)} \;=\; \frac{1}{A(1-\frac{B}{A}\cdot z)} \;=\; \frac{1}{A}\cdot\sum_{i=0}^{D-1}\left(\frac{B}{A}\right)^i\cdot z^i\,. \tag{4.4}$$

As, B^i has a trivial $\wedge\Sigma\Pi^{[\delta]}$-circuit (over $\mathsf{R}[\boldsymbol{x}]$) of size $\le 3^\delta\cdot s + i$; summing over $i\in[D-1]$, the overall size is at most $D\cdot 3^\delta\cdot s + O(D^2)$. As $D < k\cdot d$, we conclude that $1/\Psi(g)$ has $\Sigma\wedge\Sigma\Pi^{[\delta]}$ of size $\text{poly}(s\cdot k\cdot d3^\delta)$, over $\mathsf{R}[\boldsymbol{x}]$. Multiplying k-many such products directly gives an upper bound of $(s\cdot 3^\delta)^{O(k)}$, using Lemma 2.11 (basically, waring identity). □

Proof of Claim 4.9. Combining Lemma 4.12–4.13, observe that $\Psi(J_{\boldsymbol{x}_k}(\cdot)/\Psi(\cdot)$ has $\Sigma\wedge\Sigma\Pi^{[\delta]}$-circuit of size at most $(s_1\cdot s_2)^2 = (s\cdot 3^\delta)^{O(k)}$, over $\mathsf{R}[\boldsymbol{x}]$, using Lemma 2.11. Summing up at most s^k many terms (by defn. of F), the size still remains $(s\cdot 3^\delta)^{O(k)}$. □

Degree bound. As, syntactic degree of T_i are bounded by d, and Ψ maintain $\deg_{\boldsymbol{x}} = \deg_z$, we must have $\deg_z(\Psi(J_{\boldsymbol{x}_k}(g_1,\dots,g_k)) = \deg_{\boldsymbol{x}}(J_{\boldsymbol{x}_k}(g_1,\dots,g_k)) \le D-1$. Note that, Lemma 4.12 actually works over $\mathbb{F}[\boldsymbol{x},z]$ and thus there is no additional degree-blow up (in z). However, there is some degree blowup in Lemma 4.13, due to Equation (4.4).

Note that Equation (4.4) shows that over $\mathsf{R}[\boldsymbol{x}]$,

$$\frac{1}{\Psi(g)} = \left(\frac{1}{A^D}\right)\cdot\left(\sum_{i=0}^{D-1} A^{D-1-i}z^i\cdot B^i\right) =: \frac{p(\boldsymbol{x},z)}{q},$$

where $q = A^D$. We think of $p\in\mathbb{F}[\boldsymbol{x},z]$ and $q\in\mathbb{F}$. Note, $\deg_z(\Psi(g)) \le \delta$ implies $\deg_z(p) \le \deg_z((B\,z)^{D-1}) \le \delta\cdot(D-1)$.

Finally, denote $1/\Psi(g_1 \cdots g_k) =: P_{g_1,\ldots,g_k}/Q_{g_1,\ldots,g_k}$, over $\mathsf{R}[\boldsymbol{x}]$. This is just multiplying k-many (p/q)'s; implying a degree blowup by a multiple of k. In particular, – $\deg_z(P_{(\cdot)}) \leq \delta \cdot k \cdot (D-1)$ Thus, in Equation (4.3), summing up s^k-many terms gives an expression (over $\mathsf{R}[\boldsymbol{x}]$):

$$F \;=\; \sum_{g_1 \in L(T_1),\ldots,g_k \in L(T_k)} \Psi(J_{\boldsymbol{x}_k}(g_1,\ldots,g_k)) \cdot \left(\frac{P_{g_1,\ldots,g_k}}{Q_{g_1,\ldots,g_k}}\right) \;=:\; \frac{P(\boldsymbol{x},z)}{Q}\,.$$

Verify that $Q \in \mathbb{F}$. The degree of z also remains bounded by

$$\max_{g_i \in L(T_i), i \in [k]} \deg_z(P_{g_1,\ldots,g_k}) + \delta k \leq \mathrm{poly}(s).$$

Using the degree bounds, we finally have $P \in \mathbb{F}[\boldsymbol{x},z]$ as a $\Sigma \wedge \Sigma\Pi^{[\delta]}$-circuit (over $\mathbb{F}(z)$) of size $n^{O(\delta)}(s3^\delta)^{O(k)} = 3^{O(\delta k)} s^{O(k+\delta)} =: s_3$.

We want to *construct* a set $H' \subseteq \mathbb{F}^n$ such that the action $P(H',z) \neq 0$. Using [29] (Theorem 2.8), we conclude that it has $s^{O(\delta \log s_3)} = s^{O(\delta^2 k \log s)}$ size hitting set which is constructible in a similar time. Hence, the construction of Φ follows, making $\Phi(f)$ a $k+2$ variate polynomial. Finally, by the obvious degree bounds of $\boldsymbol{y}, z, t$ from the definition of Φ, we get the blackbox PIT algorithm with time-complexity $s^{O(\delta^2 k \log s)}$. This completes the proof of Theorem 1.3(b).

We could also give the final hitting set for the general problem Problem 4.6.

Problem 4.6. Let $\{T_i \mid i \in [m]\}$ be $\Pi\Sigma\Pi^{[\delta]}$ circuits of (syntactic) degree at most d and size s. Let the transcendence degree of T_i's, $\mathrm{trdeg}_{\mathbb{F}}(T_1,\ldots,T_m) = k \ll s$. Further, $C(x_1,\ldots,x_m)$ be a circuit of (size + deg) $< s'$. Design a blackbox-PIT algorithm for $C(T_1,\ldots,T_m)$.

Solution. We know that

$$C(T_1,\ldots,T_m) = 0 \iff E := \Phi(C(T_1,\ldots,T_m)) = 0.$$

Since H' can be constructed in $s^{O(\delta^2\, k\, \log s)}$-time, it is trivial to find a hitting set for $E|_{H'}$ (which is just a $k+2$-variate polynomial with the aforementioned degree bounds). The final hitting set for E can be constructed in $s'^{O(k)} \cdot s^{O(\delta^2\, k\, \log s)}$-time. □

Remark 4.14.

1. As Jacobian Criterion (Fact 4.4) holds when the characteristic is at least $d^{\,\mathrm{trdeg}}$, it is easy to conclude that our theorem holds for all fields of char $> d^k$.

2. The above proof gives an efficient reduction from blackbox PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ circuits to $\Sigma \wedge \Sigma\Pi^{[\delta]}$ circuits. In particular, a poly-time hitting set for $\Sigma \wedge \Sigma\Pi^{[\delta]}$ circuits would put PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ in P.

3. Also, DiDI-technique (of Theorem 1.1) directly gives a blackbox algorithm, but the complexity is *exponentially* worse (in terms of k in the exponent) for its recursive blowups.

4.2 PIT for $\Sigma^{[k]}\Pi\Sigma\wedge$

As we remarked earlier, the proof of Theorem 1.3(a) is similar to the one we discussed in Section 4.1. Here we sketch the proof, stating some relevant changes. Similar to Theorem 1.3(b), we generalise this theorem and prove for a much bigger class of polynomials.

Problem 4.15. Let $\{T_i \mid i \in [m]\}$ be $\Pi\Sigma\wedge$ circuits of (syntactic) degree at most d and size s. Let the transcendence degree of T_i's, $\mathrm{trdeg}_{\mathbb{F}}(T_1, \ldots, T_m) =: k \ll s$. Further, $C(x_1, \ldots, x_m)$ be a circuit of size + degree $< s'$. Design a blackbox-PIT algorithm for $C(T_1, \ldots, T_m)$.

It is trivial to see that $\Sigma^{[k]}\Pi\Sigma\wedge$ is a very special case of the above settings. We will use the same idea (and notation) as in Theorem 1.3(b), using the Jacobian technique. The main idea is to come up with Ψ map, and correspondingly the hitting set H'. If $g \in L(T_i)$, then $\mathrm{size}(g) \leq O(dn)$. The D (and hence $R[\boldsymbol{x}]$) remains as before. Claim 4.7–4.8 hold similarly. We will construct the hitting set H' by showing that F has a small $\Sigma\wedge\Sigma\wedge$ circuit over $R[\boldsymbol{x}]$.

Note that, Claim 4.10 remains the same for $\Sigma\wedge\Sigma\wedge$ (implying the same size blowup). However, Claim 4.11, the size blowup is $O(d\ \mathrm{size}(g))$, because each monomial x^e can only produce $d+1$ many monomials. Therefore, similar to Lemma 4.13, one can show that $\Psi(J_{\boldsymbol{x}_k}(g_1, \ldots, g_k)) \in \Sigma\wedge\Sigma\wedge$, of size $O(2^k k! kds)$. Similarly, the size in Lemma 4.12 can be replaced by $s^{O(k)}$. Therefore, we get (similar to Claim 4.9):

Claim 4.16. *$F \in R[\boldsymbol{x}]$ has $\Sigma\wedge\Sigma\wedge$-circuit of size $s^{O(k)}$.*

Next, the degree bound also remains the same. Following the same footsteps, it is not hard to see that while degree bound on z remains $\mathrm{poly}(ksd)$. Therefore, $P \in \mathbb{F}[\boldsymbol{x}, z]$ has $\Sigma\wedge\Sigma\wedge$-circuit of size $s^{O(k)}$.

We want to *construct* a set $H' \subseteq \mathbb{F}^n$ such that the action $P(H', z) \neq 0$. By Lemma 2.9, we conclude that it has $s^{O(k \log\log s)}$ size hitting set which is constructible in a similar time. Hence, the construction of map Φ and the theorem follows (from z-degree bound).

Solution to Problem 4.15. We know that

$$C(T_1, \ldots, T_m) = 0 \iff E := \Phi(C(T_1, \ldots, T_m)) = 0.$$

Since H' can be constructed in $s^{O(k \log\log s)}$ time, it is trivial to find a hitting set for $E|_{H'}$ (which is just a $k+2$-variate polynomial with the aforementioned degree bounds). The final hitting set for E can be constructed in $s'^{O(k)} \cdot s^{O(k \log\log s)}$ time. □

5 Conclusion

This work introduces the powerful DiDI-technique and solves three open problems in PIT for depth-4 circuits, namely $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$ (blackbox) and $\Sigma^{[k]}\Pi\Sigma\wedge$ (both whitebox and blackbox). Here are some immediate questions of interest which require rigorous investigation.

1. Can the exponent in Theorem 1.1 be improved to $O(k)$? Currently, it is exponential in k.

2. Can we improve Theorem 1.3(b) to $s^{O(\log\log s)}$ (like in Theorem 1.3(a))?

3. Can we design a polynomial-time PIT for $\Sigma^{[k]}\Pi\Sigma\Pi^{[\delta]}$?

4. Can we design a polynomial time PIT for $\Sigma\wedge\Sigma\Pi^{[\delta]}$ circuits (that is, unbounded top-fanin)?

5. Can we solve PIT for $\Sigma^{[k]}\Pi\Sigma\mathsf{M}_2$ circuits efficiently (polynomial/quasipolynomial-time), where $\Sigma\mathsf{M}_2$ denotes bivariate polynomials?

6. Can we design an efficient PIT for rational functions of the form $\Sigma\,(1/\Sigma\wedge\Sigma)$ or $\Sigma\,(1/\Sigma\Pi)$ (for *un*bounded top-fanin)?

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